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# States of One-Dimensional Coulomb Systems as Simple Examples of $\theta$ Vacua and Confinement

**Michael Aizenman**[1] **and Jürg Fröhlich**[2]

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The one-dimensional Coulomb system is known to have equilibrium states with nonvanishing electric field. These states are shown here to be analogous, and related, to the $\theta$ vacua which have been discussed for gauge theories in two or more space-time dimensions. The system exhibits confinement of fractional charges, which we dicuss with the purpose of offering a simple example of the $\theta$-vacua phenomenology. Precise relations and connections between one-dimensional Coulomb gases and two-dimensional Abelian gauge theories, and quantum-mechanical matter systems, are discussed.

**KEY WORDS:** Coulomb systems; $\theta$ vacua; confinement; one-dimensional model.

## 1. INTRODUCTION AND SUMMARY OF RESULTS

It has been recently shown that, at fixed values of the thermodynamic parameters, the one-dimensional Coulomb gas has a one-parameter family of equilibrium states with a generally nonzero expectation value of the electric field.(1) In that paper a point of view was adopted which offers a unified understanding of the occurrence of those states and of the breaking of translation symmetry in the one-dimensional jellium model, due to the formation of a Wigner lattice.(2,3)

In this note we show that the existence of a one-parameter family of equilibrium states in the one-dimensional Coulomb gas is a phenomenon closely related to that of the well-known $\theta$ vacua in gauge field theories on

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[1] Department of Physics, Princeton University, Princeton, New Jersey 08540, USA.
[2] Institut des Hautes Études Scientifiques, 35, Route de Chartres, F-91440 Bures-sur-Yvette, France.



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a space-time of dimension 2 or more.[4] Furthermore we describe an analogy between the behavior of fractional charge correlations in the Coulomb gas and that of the expectation of a fractionally charged Wilson loop in two-dimensional Abelian gauge theories.

The Coulomb gas may be analyzed both from a point of view of statistical mechanics, including an approach based on the *electric field ensemble*,[5] and a more field theoretical point of view involving the *sine-Gordon* (*Siegert*) *representation*.[6] This second approach serves to exhibit close connections between the one-dimensional Coulomb gas and two-dimensional Abelian Higgs models. In both models one finds a family of $\theta$ states and "confinement" of fractional charges. The first point of view may serve to interpret those phenomena, exhibited here in what may be their simplest manifestation, in a language more familiar to the expert in statistical mechanics.

The purpose of our note is primarily a *pedagogical* one. The occurrence of the $\theta$-states phenomenology in a one-dimensional system, as well as most of the connections and analogies which we point out, may be well known to experts, in one way or another.

We conclude the introduction with a brief summary of the sections of this note. In Section 2 we review the electric field ensemble, and reconsider the $\theta$ states and the $\theta$ dependence of the free energy in the one-dimensional Coulomb gas. In Section 3 we recall the sine-Gordon representation for the one-dimensional Coulomb gas, discuss the connection with the Mathieu equation, and present a construction and analysis of $\theta$ states in the sine-Gordon representation. This representation is very useful for investigating "instanton effects" and comparing them with exact results. In Section 4, properties of fractional charge correlations and their $\theta$ dependence are studied. It is shown that, in the $\theta = 0$ state, two opposite fractional charges, immersed in the system, feel an approximately constant attractive Coulomb force ("confinement of fractional charges"). In other $\theta$ states the force can be attractive *or* repulsive—depending on $\theta$ and the value of the fractional charge. In the latter case the fractional charges are expelled to screen charged sources at $\pm\infty$. Section 4 is concluded with some comments on the screening properties and the decay rate of more general correlations in the one-dimensional Coulomb gas (absence of exponential Debye screening). In Section 5 we compare the results of previous sections with analogous, and known, results for Abelian gauge theories (quantum electrodynamics) in two space-time dimensions. We also point out some differences. Section 6 contains some conclusions and open problems. Among them a somewhat interesting one concerns the properties of a many-component one-dimensional Coulomb gas with irrationally related charges, for which we suggest a possibility of a phase transition.

Although that gas is quite unphysical it can perhaps be used to study properties of one-dimensional Schrödinger operators with quasiperiodic potentials.

## 2. $\theta$ STATES OF THE ONE-DIMENSIONAL COULOMB GAS

The Coulomb interaction in one dimension is described by the potential energy

$$V(\{\sigma_i, q_i\}) = -\tfrac{1}{2}\sum_{i,j}\sigma_i\sigma_j|q_i - q_j| \tag{2.1}$$

of a configuration of charges, $\sigma_i$, located at positions $q_i \in \mathbb{R}$; $V(x) \equiv -\sum_i \sigma_i |x - q_i|$ satisfies the Poisson equation

$$-\frac{d}{dx^2} V(x) = 2\sum_i \sigma_i \delta(x - q_i) \tag{2.2}$$

For a *neutral* gas of charges $\sigma_i = \pm e$ (confined to an interval $[-L, L]$), one defines the partition function, $\Xi_L(\beta, z)$, and computes the "free energy," $P_0(\beta, z)$, as follows:

$$\Xi_L(\beta, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\substack{\sigma_i = \pm e \\ \Sigma\sigma_i = 0}} \int_{-L}^{L} \cdots \int dq_1 \cdots dq_n \exp[-\beta V(\{\sigma_i, q_i\}_{i=1,\ldots,n})] \tag{2.3}$$

$$P_0(\beta, z) = \lim_{L\to\infty} \frac{-1}{2L\beta} \ln \Xi_L(\beta, z) \tag{2.4}$$

Here $z$ is the fugacity, which is taken to be equal for both species of charges, and $\beta$ is the inverse temperature.

$P_0(\beta, z)$ was found in Lenard[7] and Prager[8] to be

$$P_0(\beta, z) = \inf \operatorname{spec}\left[-e^2\beta \frac{d^2}{dx^2} + 2z(\cos x + 1); \quad \text{on} \quad L^2(\mathbb{R})\right] \tag{2.5}$$

The corresponding spectral problem is the Mathieu equation, which is well studied.[9]

The above result was obtained in Ref. 7 by the analysis of neutral clusters. However, as was pointed out and used in subsequent papers of Edwards and Lenard[6] and Lenard,[5] the Coulomb systems may be analyzed from other points of view. One (discussed in the next section) involves the sine-Gordon transformation, and yet another one results from focusing the attention on the electric field, $E(x) = -dV(x)/dx$, as the basic variable describing the system.[5] The latter point of view was used in Aizenman and Martin[1] to show that a neutral Coulomb system admits a one-parameter family of distinct equilibrium states, obtained by placing the system in a constant electric field. In the remainder of this section we shall briefly summarize the argument.

At a constant external field, $D$, the total energy of a system of charges in the interval $[-L, L]$ is

$$H = V(\{\sigma_i, q_i\}) - \Big(\sum_i \sigma_i q_i\Big) D \tag{2.6}$$

$H$ may be expressed as a functional of the total electric field, $E(x) = D + \sum_i \sigma_i \operatorname{sgn}(x - q_i)$, by the following electrostatic identity:

$$H = \frac{1}{4}\int_{-L}^{L} dx\, |E(x)|^2 - \frac{L}{2}\Big(\sum_i \sigma_i\Big)^2 \tag{2.7}$$

The electric field in this system is piecewise constant and has jump discontinuities, of magnitude $\pm 2e$, at the positions of the charges. Let us denote by $\nu(dE)$ the measure, on the space of all such electric field configurations, characterized by the following two conditions (which are equivalent for all values of $x_0 \in \mathbb{R}$):

(i) $\nu(\{E(\cdot)\,|\,E(x_0) = u\}) = 1, \qquad \forall u \in \mathbb{R}$.

(ii) For a given value of $E(x_0)$, the subsequent jumps, by $\pm 2e$, of $E(x)$ as a function of $x$ are mutually independent and occur with equal densities $z$.

It may be easily seen[1,5] that the measure on the electric field ensemble, induced by the a priori distribution of charges [which is described by the summation and integration in the right-hand side of (2.3)] is $\delta_{E(-L),E(L)}\delta(E(-L))\nu(dE)$. The first factor is a Kronecker $\delta$ and reflects the neutrality of the ensemble. The seond is a Dirac $\delta$ function and must be replaced by $\delta(E(-L) - D)$ when there is an external field $D$.

The partition function in the external field may now be expressed as follows:

$$\Xi_L(\beta, z; D) = \int \nu(dE)\,\delta_{E(-L),E(L)}\,\delta(E(-L) - D) \times \exp\left[-\frac{\beta}{4}\int_{-L}^{L} dx\,|E(x)|^2\right] \tag{2.8}$$

It may be shown that large fluctuations in $E(\cdot)$ are suppressed by the Gibbs factor in (2.8), and that in the thermodynamic limit, $L \to \infty$, $E(\cdot)$ attains a limiting distribution.[5,1] (This is not the case for the electric potential.) In order to understand the dependence of this distribution on $D$, one is advised to consider the Coulomb screening in one dimension, and its limitations. When an external field is applied, it is screened to a large extent by an accumulation of charges near the boundary. However, screening is in general incomplete since the field produced by a charge is constant. This is drastically reflected in the fact that the range of values of $E(\cdot)$ is given by

$$E(x) \in D + 2e\mathbb{Z}, \qquad \forall x \in \mathbb{R} \tag{2.9}$$

Thus the limiting state for $E(\cdot)$ has a nontrivial dependence on $\theta = D_{\text{mod. } 2e}$.

The above structural argument does not yet show that in the different limiting states for $E(\cdot)$ the electric charges are distributed differently, since two electric field configurations which differ only by an overall constant correspond to the same charge configuration. However, if one can show that for "typical" configurations in the above limiting states the *total* electric field can, in some way, be reconstructed from the positions of the charges, then it would follow that the charge distribution depends on $D$. This is the approach of Ref. 1, where the following statements were proven.

**Proposition 1.** (i) For each value of $D$, $E(\cdot)$ has a limiting probability distribution which depends only on $\theta = D_{\text{mod. } 2e}$. Furthermore, these "$\theta$ states" are Markovian and translation invariant in $x$, and satisfy (2.9).

(ii) The average field in the $\sigma$ states, $\langle E\rangle_\theta$, does not vanish except for $\theta = 0, e$.

(iii) The following relations are satisfied with probability 1 in each $\theta$ state:

$$E(x) = -\lim_{r\to\infty} 2 \sum_{q_i\in[x,x+r]} \sigma_i\left(1 - \frac{q_i - x}{r}\right) + \langle E\rangle_\theta$$

$$= \lim_{r\to\infty} 2 \sum_{q_i\in[x-r,x]} \sigma_i\left(1 + \frac{q_i - x}{r}\right) + \langle E\rangle_\theta \tag{2.10}$$

(iv) In each $\theta$ state, with probability 1,

$$\lim_{r\to\infty} \exp\left[i2\pi \sum_{q_i\in[0,r]} \left(\frac{q_i}{r}\right)\left(\frac{\sigma_i}{e}\right)\right] = \exp\left[i2\pi g(\theta)\right] \tag{2.11}$$

where $g(\cdot)$ is a strictly increasing function of $\theta$. ■

Equation (2.11), derived by exponentiating (2.10), is an explicit characterization of typical charge configurations, distinguishing the various $\theta$ states. Other, somewhat surprising features of (2.10), are discussed in Ref. 1.

The free energy of a neutral Coulomb gas in an external field $D$ can also be calculated from (2.8). By an application of the Feynman–Kac formula, the partition function may be rewritten as the kernel of an operator on $l^2(\theta + 2e\mathbb{Z})$, which is the transfer matrix for $E(\cdot)$ [constrained by (2.9)], namely

$$\Xi_L(\beta, z; D) = \exp\left[-2L\left(-z\Delta + \tfrac{1}{4}\beta\tilde{E}^2\right)\right](-L, L) \tag{2.12}$$

Here $\Delta$ is the discrete Laplacian, defined by

$$(\Delta f)(u) = f(u+2e) + f(u-2e) - 2f(u) \tag{2.13}$$

and $\tilde{E}$ is a multiplication operator, given by

$$(\tilde{E}f)(u) = uf(u) \tag{2.14}$$

The "free energy" is now easily seen to be

$$\begin{aligned} P_\theta(\beta, z) &= \lim_{L\to\infty} \frac{-1}{2L} \ln \Xi_L(\beta, z; D) \\ &= \inf \operatorname{spec}\Big(\big(-z\Delta + \tfrac{1}{4}\beta\tilde{E}^2\big); \quad \text{on} \quad l^2(\theta + 2e\mathbb{Z})\Big) \end{aligned} \tag{2.15}$$

In the Fourier-transformed representation, on $L^2([-\pi,\pi])$, the above operator becomes $e^2\beta(id/dx + \theta/2e)^2 + 2z(\cos x + 1)$. The unitary transformation by $U = e^{-ix\theta/2e}$ transforms this operator to $-e^2\beta\Delta_\theta + 2z(\cos x + 1)$, where the Laplacian $\Delta_\theta$ is defined with the boundary conditions $f(-\pi) = f(\pi)e^{i2\pi\theta/2e}$. Therefore (2.15) may be rephrased as follows.

**Proposition 2.** The free energy which corresponds to the $\theta$ states, $P_\theta(\beta, z)$, is given by the lowest-band eigenvalue of $-e^2\beta\Delta + 2z(\cos x + 1)$ [on $L^2(\mathbb{R})$], at the Bloch momentum $k = \theta/2e$. ■

In the next section we shall discuss the $\theta$ states from a field theoretic point of view.

## 3. THE COULOMB GAS IN THE SINE-GORDON REPRESENTATION

### 3.1. Sine-Gordon Representation

The sine-Gordon representation offers a powerful approach to the study of systems with pair interations of positive type, including Coulomb systems (with a neutrality condition in one and two dimensions). Its application to one-dimensional Coulomb systems first appeared in the work of Edwards and Lenard.[6]

Let $\rho(x) = \sum \sigma_i \delta(x - q_i)$ denote the charge density. The Gibbs factor in (2.3), with the imposed neutrality condition, can be expressed by the following functional integral:

$$\begin{aligned} &\exp\left[ -\frac{\beta}{2} \int\int dx\, dy\, \rho(x)|x-y|\rho(y) \right] \cdot \delta_{\int dx\, \rho(x), 0} \\ &\quad = \left\langle\left\langle \exp\left[ i \int dx\, \rho(x)\phi(x) \right] \right\rangle\right\rangle \end{aligned} \tag{3.1}$$

Here $\langle\langle - \rangle\rangle$ represents an average over Brownian paths, $\phi(x)$, with $x$ as "time" parameter, whose initial point, $\phi(0)$, is uniformly averaged over $\mathbb{R}$. Alternatively, $\langle\langle - \rangle\rangle$ may be regarded as the limiting normalized expectation for a Gaussian random field $\phi_\epsilon(x)$ with covariance $(\beta(-\Delta)^{-1} + \epsilon^2)^{-1}$,

$\epsilon \downarrow 0$, and the Gibbs factor may be viewed as

$$\exp\left[-\frac{\beta}{2}\int\int dx\,dy\,\rho(x)(-\Delta)^{-1}\rho(y)\right]$$

$(-\Delta)^{-1}$ being the quadratic form which is $+\infty$, unless $\int dx\,\rho(x)=0$.

Equation (3.1) leads to the following expression for the partition function:

$$\Xi_L(\beta,z)=\left\langle\left\langle \exp\left\{2z\int_{-L}^{L}dx\cos\left[e\phi(x)\right]\right\}\right\rangle\right\rangle \tag{3.2}$$

The correlation functions have a related expression,

$$\rho_n^{(L)}(\{\sigma_i,q_i\})=\left\langle\prod_{j=1}^{n}\exp\left[i\sigma_j\phi(q_j)\right]\right\rangle^{(L)} \tag{3.3}$$

where

$$\langle -\rangle^{(L)}=$$

$$\left\langle\left\langle -\exp\left\{2z\int_{-L}^{L}dx\cos\left[e\phi(x)\right]\right\}\right\rangle\right\rangle \Big/ \left\langle\left\langle \exp\left\{2z\int_{-L}^{L}dx\cos\left[e\phi(x)\right]\right\}\right\rangle\right\rangle \tag{3.4}$$

One convenience of the above formalism lies in the Markov property of $\phi$, which was used by Edwards and Lenard[6] to prove the existence of the limit for correlation functions. It also lends itself to an application of correlation inequalities, used in Fröhlich and Park[10] to prove the existence and monotonicity properties of the correlation functions for a wide class of systems in arbitrary dimension.

The state of the field $\phi(\cdot)$ described by $\langle -\rangle=\lim_{L\to\infty}\langle -\rangle^{(L)}$, can be represented by the formal measure

$$\text{“}\prod_x d\phi(x)\exp\left\{-\int dx\left[\frac{\beta^{-1}}{2}|\nabla\phi|^2-2z\cos(e\phi(x))\right]\right\}/\text{norm”} \tag{3.5}$$

Thus, in addition to the local fluctuations, which resemble a Brownian motion, $\phi$ has a bias towards the values $\phi=2\pi n/e$, $n\in\mathbb{Z}$. Similarly to $\langle\langle -\rangle\rangle$, the state $\langle -\rangle$ does not tie down $\phi$, and involves an average over periodic shifts. However, its restriction to periodic functions of $\phi$, with period $2\pi/(\sqrt{\beta}\,e)$, corresponds to a well-defined positive probability distribution. In particular, (3.3) holds also in the limit $L\to\infty$.

The variational equation derived from the action in (3.5),

$$-\beta^{-1}\Delta\phi=2ze\sin\left[e\phi(x)\right]=0 \tag{3.6}$$

has an "instanton" ("anti-instanton") solution with boundary values $\phi(-\infty)=0$, $\phi(+\infty)=+2\pi/e$. One may expect that in the state $\langle - \rangle$ the field $\phi$ typically fluctuates in the vicinity of functions which locally minimize the action, resembling a gas (i.e., a superposition) of instantons and anti-instantons (with some interaction).

To summarize, the field $\phi(\cdot)$ is conjugate, via Fourier transformation, to the charge density field $\rho(\cdot)$. The distribution of $\phi$ is governed by an action with a term $-(\beta^{-1}/2)\int dx\,|\nabla\phi|^2$ coming from the Gibbs factor, and a term $\int dx\, 2z\cos(e\phi)$ corresponding to the a priori distribution of the charges. That the second term is real is a consequence of charge symmetry. The positivity of the measure for $\phi$ is a rather powerful tool in deriving various properties of a more general class of systems.[11]

### 3.2. $\theta$ States

Having introduced the "vacuum state" $\langle - \rangle$ corresponding to the neutral Coulomb gas, we shall now construct what corresponds to the $\theta$ states discussed in Section 2. We shall show in Section 5 that this is an analogous construction to that of the $\theta$ vacua which were widely discussed in models with gauge fields in higher dimensions.[4]

In order to place the neutral gas in an external field $D=\theta$, one may put a fixed pair of charges $\pm\theta/2$ at opposite ends of the system. Equation (3.1) implies therefore that the correlation functions in the $\theta$ states are given by the following limits:

$$\rho_n^{(\theta)}(\{\sigma_i, q_i\}) = \lim_{L\to\infty} \left\langle \prod_{i=1}^{n} e^{i\sigma_i\phi(q_i)} \right\rangle_\theta^{(L)} \tag{3.7}$$

where

$$\langle - \rangle_\theta^{(L)} = \left\langle\left\langle - \exp\left[i\frac{\theta}{2}\phi(L)\right]\exp\left[-i\frac{\theta}{2}\phi(-L)\right]\exp\left\{2z\int_{-L}^{L}dx\cos\left[e\phi(x)\right]\right\}\right\rangle\right\rangle \Big/ \left\langle\left\langle \exp\left[i\frac{\theta}{2}\phi(L)\right]\exp\left[-i\frac{\theta}{2}\phi(-L)\right]\exp\left\{2z\int_{-L}^{L}dx\cos\left[e\phi(x)\right]\right\}\right\rangle\right\rangle \tag{3.8}$$

The $\theta$ states correspond to the states

$$\langle - \rangle_\theta = \lim_{L\to\infty} \langle - \rangle_\theta^{(L)} \tag{3.9}$$

whose statistical mechanical interpretation is similar to that of the state $\langle - \rangle \equiv \langle - \rangle_0$; see (3.3), (3.4). Formally,

$$\exp\left[i\frac{\theta}{2}\phi(L)\right]\exp\left[-i\frac{\theta}{2}\phi(-L)\right] = \exp\left[i\frac{\theta}{2}\int_{-L}^{L}dx\,\frac{d}{dx}\phi(x)\right] \tag{3.10}$$

Therefore, the modification of the action density which yields the state $\langle - \rangle_\theta$ is the addition of the term $i\frac{1}{2}\theta d\phi(x)/dx$. Note that the random field $-i(d/dx)\phi(x)$ corresponds to the *electric field*,[6] e.g.,

$$\left\langle -i\frac{d}{dx}\phi(x)\right\rangle_\theta = \langle E(x)\rangle_\theta = 2\beta\frac{d}{d\theta}P_\theta(\beta,z) \tag{3.11}$$

The integral

$$\frac{e}{2\pi}\int dx\frac{d}{dx}\phi(x)$$

is also the "instanton charge," i.e., counts the number of instantons minus the number of anti-instantons. Thus, (3.10) has the significance of assigning a complex fugacity to the instantons, and its conjugate to the anti-instantons, in the "instanton gas." (Incidentally, the instanton solutions of the Mathieu equation are of course the time-independent soliton solutions of the classical sine-Gordon equation.)

## 4. "CONFINEMENT" IN $\theta$ STATES

An interesting feature of the $\theta$ states is the "confinement" of fractional charges, which in Section 5 will be related to the confinement bounds for the "Wilson loop"; see Ref. 12.

Let us consider the equilibrium distribution of a pair of charges $\pm\alpha$ inserted into a neutral Coulomb gas. We fix the charge $+\alpha$ at the origin. If the Coulomb gas is in one of the $\theta$ states then, using the sine-Gordon representation discussed in the previous section, the equilibrium distribution of the change $-\alpha$ is determined by the following function:

$$P_{\alpha;\theta}(x) = \langle e^{i\alpha\phi(0)}e^{-i\alpha\phi(x)}\rangle_\theta \tag{4.1}$$

There are now two qualitatively different possibilities:

(i) If $\int_{-\infty}^{\infty}dx\,\rho_{\alpha;\theta}(x)=\infty$, then the charge $-\alpha$ would drift away and would not be observed in any finite region.

(ii) If $\int_{-\infty}^{\infty}dx\,\rho_{\alpha;\theta}(x)<\infty$, then the charge $-\alpha$ has a probability distribution on the line, whose density is $\rho_{\alpha;\theta}(x)/\int_{-\infty}^{\infty}dy\,\rho_{\alpha;\theta}(y)$.

In the second case, the charges exhibit "confinement." Such a situation would not be observed in systems with short-range interactions.

The asymptotic behavior of $\rho_{\alpha;\theta}$ is described by the following result.

**Proposition 3.** For any $\theta\in[0,2e)$, $\alpha\in\mathbb{R}$

$$\lim_{\substack{x\to+\infty\\(-)}}\frac{-1}{|x|}\ln\langle e^{i\alpha\phi(0)}e^{-i\alpha\phi(x)}\rangle_\theta = P_{\substack{\theta+2\alpha\\(-)}}(\beta,z)-P_\theta(\beta,z) \tag{4.2}$$

extending the definition of $P_\theta$ periodically in $\theta$.

The proof of (4.2) involves a rather straightforward application of the transfer-matrix formalism and is not given here in full detail. Let us remark that (4.2) can also be shown using the electric field formulation, where $\rho_{\alpha;\theta}(x)$ is obtained by an integral similar to (2.8). However, the electric field is now constrained by

$$E(y) \in \begin{cases} \theta + 2e\mathbb{Z}, & y \notin [0,x] \\ \theta + 2\alpha \operatorname{sgn} x + 2e\mathbb{Z}, & y \in [0,x] \end{cases} \tag{4.3}$$

In fact, the transfer matrix formalism leads to the stronger conclusion, that

$$\lim_{\substack{x \to +\infty \\ (-)}} \left\{ \ln\langle e^{i\alpha\phi(0)} e^{-i\alpha\phi(x)}\rangle - |x| \Big[ P_{\theta \underset{(-)}{+} 2\alpha}(\beta, z) - P_\theta(\beta, z) \Big] \right\} = C_{\underset{(-)}{+}} \tag{4.4}$$

for some, explicitly known, constants $C_+, C_- \in (0,1]$. In addition to the asymptotics described in Proposition 3 and (4.4) we have the following upper bounds.

**Proposition 4.** For arbitrary $\theta \in [0, 2e)$, $\alpha \in \mathbb{R}$,

$$\langle e^{i\alpha\phi(0)} e^{-i\alpha\phi(x)}\rangle_\theta \leqslant \begin{cases} \exp\{-|x|[P_{\theta+2\alpha}(\beta,z) - P_\theta(\beta,z)]\}, & x > 0 \\ \exp\{-|x|[P_{\theta-2\alpha}(\beta,z) - P_\theta(\beta,z)]\}, & x < 0 \end{cases} \tag{4.5}$$

***Proof.*** It follows from the general theory of reflection positivity,[13,14] or from the existence of a self-adjoint transfer matrix, that the states $\langle - \rangle_\theta$ are reflection positive; i.e., for all functons $F$ depending only on $\{\phi(x): x \in [0,\infty)\}$,

$$\langle \overline{F(R\phi)}\, F(\phi) \rangle_\theta \geqslant 0 \tag{4.6}$$

where $(R\phi)(x) \equiv \phi(-x)$. [Inequality (4.6) can also be deduced from the Markov and reflection properties of $\langle - \rangle_\theta$; see, e.g., Ref. 15.]

By (4.6), we can apply the chessboard estimate[13] which yields

$$\begin{aligned} \langle e^{i\alpha[\phi(0)-\phi(x)]}\rangle_\theta &\leqslant \left\langle \prod_{k=0}^{n-1} e^{i\alpha[\phi(kx)-\phi((k+1)x)]} \right\rangle_\theta^{1/n} \\ &= \langle e^{i\alpha[\phi(0)-\phi(nx)]}\rangle_\theta^{1/n} \end{aligned} \tag{4.7}$$

from which Proposition 4 follows by taking $n \to \infty$ and applying (4.2). Another, basically equivalent proof follows directly from the existence of a self-adjoint transfer matrix and the spectral theorem which imply that $\langle e^{i\alpha[\phi(0)-\phi(x)]}\rangle^{1/|x|}$ is increasing in $|x|$. ∎

Proposition 2 relates $P_\theta(\beta, z)$ to a well-studied function of $\theta$.[9] It has its minimum at $\theta = 0$, is strictly increasing in $\theta$ on $[0, e]$ and is symmetric about $\theta = e$. These properties together with Proposition 4 imply "confinement" of charges which are *fractional*, in units of $e$, in the $\theta = 0$ state. In the $\theta = e$ state, the second charge is always expelled to infinity, since $P_{e+2\alpha}(\beta, z) - P_e(\beta, z)$ is negative for $\alpha \neq 0$, mod $e$. In other states we will generally observe both types of behavior, depending on the value of $\alpha$. Moreover, for suitable $\theta$ and $\alpha$ the second charge is confined in $\{x > 0\}$ and expelled in $\{x < 0\}$ (or vice versa).

We conclude this section with some comments on the cluster properties of the standard equilibrium state, $\langle - \rangle_{\theta=0}$, of the Coulomb gas. It might be tempting to interpret the exponential decay of the fractional charge correlation $\langle e^{i\alpha\phi(0)} e^{-i\alpha\phi(x)} \rangle_0$, as $|x| \to \infty$, as some sort of screening. This interpretation is not correct, and the behavior of the fractional charge correlation really shows that the electric field produced by a fractional charge cannot be shielded by the charges belonging to the ensemble. The opposite situation occurs in the two- or higher-dimensional Coulomb gas (with a Coulomb potential regularized at short distances), for which Brydges[16] has proven exponential Debye screening, at small $\beta$ and moderate value of $z$. It follows from his results that

$$\langle e^{i\alpha[\phi(0) - \phi(x)]} \rangle \xrightarrow[|x| \to \infty]{} M(\beta, z)^2 > 0 \tag{4.8}$$

if $\beta$ and $z$ are chosen such that Debye screening holds; see also Ref. 11. Thus, even the logarithmic Coulomb potential between two fractionally charged sources immersed in a two-dimensional Coulomb gas can be shielded by the charges in the ensemble, provided $\beta$ is small enough and $z > 0$ is suitable. In three or more dimensions, (4.8) is true for arbitrary $\beta$ and $z$, in two dimensions it holds *only* in the *plasma phase* of the gas with Debye screening.[11] Finally, we show that in the one-dimensional Coulomb gas there are correlations with an arbitrarily slow power falloff, for all values of $\beta$ and $z$, so that there is no exponential Debye screening. For this purpose, we choose two sequences, $\{\alpha_n\}$ and $\{C_n\}$, with the properties $\alpha_n \neq \alpha_m$ for $n \neq m$, $\alpha_n \to 0$ as $n \to \infty$, and $\sum_{n=0}^{\infty} |C_n|^2 < \infty$, and we define an observable

$$A(\phi) = \sum_{n=0}^{\infty} C_n e^{i\alpha n\phi}$$

Since, by the neutrality condition,

$$\langle e^{i\alpha\phi(0)} e^{-i\alpha'\phi(x)} \rangle_0 = 0$$

for fractional $\alpha \neq \alpha'$ we have

$$\begin{aligned} \langle A(\phi(0)) \overline{A(\phi(x))} \rangle &= \sum_{n=0}^{\infty} |C_n|^2 \langle e^{i\alpha_n[\phi(0) - \phi(x)]} \rangle_0 \\ &\geqslant \sum_{n=0}^{\infty} |C_n|^2 e^{-\frac{1}{2}\beta\alpha_n^2|x|} \end{aligned} \tag{4.9}$$

The last inequality follows by Jensen's inequality.[11] By choosing $\{\alpha_n\}$ and $\{C_n\}$ suitably, $\langle A(\phi(0))\overline{A(\phi(x))}\rangle$ may be given an arbitrarily slow power falloff.

## 5. THE ANALOGY BETWEEN THE ONE-DIMENSIONAL COULOMB GAS AND TWO-DIMENSIONAL QUANTUM ELECTRODYNAMICS

In this section we recall some analogies and a connection between the one-dimensional Coulomb gas and quantum electrodynamics in two space-time dimensions, describing charged matter fields $\phi$ (scalar), $\Psi$ (Dirac spinor), . . . with electromagnetic interactions. We adopt the Euclidean description of relativistic quantum field theory.[17]

In two-dimensional space-time all matter currents built from charged scalar and Dirac spinor fields are functionals of scalar Bose fields $\phi, \chi, \ldots$; see Refs. 18, 19. Thus, as long as we do not wish to describe charged sectors, the theory can be described entirely in terms of scalar Bose fields and the electromagnetic vector potential, $A_\mu$, which, however, could be eliminated by explicit integration. This observation permits one to analyze two-dimensional quantum electrodynamics—even spinor QED—in terms of Euclidean functional integrals. Subsequently, we will make use of this simplification—mainly in order to simplify the exposition.

The interaction between matter and the electromagnetic field is described by the term

$$ie\int J_\mu(x,t)A^\mu(x,t)\,dx\,dt \tag{5.1}$$

in the total Euclidean action $\mathscr{Q}(\phi,\chi,\ldots;A_\mu)$, where $e$ is the electric charge and

$$J_\mu = j_\mu^\phi + j_\mu^\chi + \cdots \tag{5.2}$$

is the total Euclidean electromagnetic current, with $j_\mu^\phi$, $j_\mu^\chi$ the contributions coming from $\phi$ and $\chi$, respectively. The Euclidean vacuum expectation is given by the formal functional measure

$$d\mu_0(\phi,\chi,\ldots;A_\mu) = \text{“}Z^{-1}e^{-\mathscr{Q}(\phi,\chi,\ldots;A_\mu)}\prod_x d\phi(x)\,d\chi(x)\cdots\prod_{x,\mu}dA_\mu(x)\text{”} \tag{5.3}$$

The mathematical existence of measures $d\mu(\phi,\chi,\ldots;A_\mu)$ for the models mentioned above has been verified in Refs. 19, 20 (spinor QED) and in Ref. 21 (scalar QED, or, respectively, Higgs model). From the functional measure (5.3) the theory on the vacuum sector can be reconstructed.[17]

We now analyze the effect of coupling, in addition to the quantized matter fields, a classical current, $j_\psi^{\text{class}}$, describing two $c$-number charged sources at spatial $\pm\infty$ to the electromagnetic field. This contributes an additional term

$$\delta \mathcal{Q} = i \int j_\mu^{\text{class}}(x,t) A^\mu(x,t)\, dx\, dt \tag{5.4}$$

to the Euclidean action $\mathcal{Q}$. The current $j_\mu^{\text{class}}$ is conserved, hence it satisfies the continuity equation

$$\partial^\mu j_\mu^{\text{class}} = 0 \tag{5.5}$$

Since space-time is two-dimensional, the solutions of equation (5.5) have the form

$$j_\mu^{\text{class}} = -\epsilon_{\mu\nu}\partial^\nu f \tag{5.6}$$

where $f$ is a scalar function, and $\epsilon_{\mu\nu} = 0$ for $\mu = \nu$, $\epsilon_{\mu\nu} = \text{sign}(\mu, \nu)$ otherwise. We now suppose that $j_\mu^{\text{class}}$ is the current corresponding to two classical sources with fractional charge $\pm\theta/2$, placed at $x = \pm L$, which appear at (imaginary) time $t = -T$ and disappear at $t = T$. Hence the support of $j_\mu^{\text{class}}$ is the loop $\mathcal{L}_{L\times T}$ which is the boundary of the rectangle

$$L \times T \equiv \{(x,t) : |x| \leqslant L, |t| \leqslant T\}$$

Since the charges of the classical sources are $\pm\theta/2$, the correct choice of the function $f$ appearing in (5.6) is

$$f(x,t) = \tfrac{1}{2}\theta\chi_{L\times T}(x,t)$$

where $\chi_{L\times T}$ is the characteristic function of $L \times T$. By (5.4) and (5.6)

$$\begin{aligned} \delta\mathcal{Q} \equiv \delta\mathcal{Q}^\theta_{L\times T} &= i\theta/2 \oint_{\mathcal{L}_{L\times T}} A_\mu(\xi)\, d\xi^\mu \\ &= i\theta/2 \int F(x,t)\chi_{L\times T}(x,t)\, dx\, dt \end{aligned} \tag{5.7}$$

where $F = \text{curl} A$ is the Euclidean electromagnetic field strength. As in Section 3, (3.8)–(3.10), we now introduce a perturbed functional measure, giving rise to a $\theta$ vacuum:

$$\begin{aligned} &d\mu_\theta(\phi, \chi, \ldots; A_\mu) \\ &= \lim_{L,T\to\infty} \left(Z^\theta_{L\times T}\right)^{-1} e^{-\delta\mathcal{Q}^\theta_{L\times T}} d\mu_0(\phi,\chi,\ldots; A_\mu) \\ &= \lim_{L,T\to\infty} \left(Z^\theta_{L\times T}\right)^{-1} \exp\left[-i\frac{\theta}{2}\oint_{\mathcal{L}_{L\times T}} A_\mu(\xi)\times d\xi^\mu\right] d\mu_0(\phi,\chi,\ldots; A_\mu) \end{aligned} \tag{5.8}$$

Indeed, this formula is strictly analogous to formulas (3.8)–(3.10) for the $\theta$ state of the one-dimensional Coulomb gas (respectively, the imaginary-time description of the Bloch electron with momentum $\pi\theta/e$). In analogy to that case, the term $(e/2\pi)\oint A_\mu(\xi)\,d\xi^\mu$ can be interpreted as counting the difference between the number of instantons and the number of anti-instantons. In the case of the two-dimensional Higgs model the instantons are the vortices. The classical field equations do have vortex solutions, and the magnetic flux of these solutions is "quantized":

$$\lim_{L,T\to\infty} \oint_{\mathcal{C}_{L\times T}} A_\mu(\xi)\,d\xi^\mu = \int_{\mathbb{R}^2} F(x,t)\,dx\,dt = \frac{2\pi}{e}\,n, \qquad n\in\mathbb{Z}$$

Thus

$$\lim_{L,T\to\infty} \exp\left[\frac{i\theta}{2}\oint_{\mathcal{C}_{L\times T}} A_\mu(\xi)\,d\xi^\mu\right]$$

is periodic in $\theta$ with period $2e$. We may therefore expect that $d\mu_\theta(\phi,\chi,\ldots;A_\mu)$ is periodic in $\theta$ with period $2e$, too. This property has been established for spinor QED, in bosonized form, in Refs. 22, 19 and for scalar QED or, respectively, the Higgs model on a two-dimensional lattice in Ref. 21. In both cases the physics depends nontrivially on $\theta$. For $\theta\neq 0, e$, for example, the expectation of the electric field in a $\theta$ vacuum is nonzero,[22,19] as in the one-dimensional Coulomb gas. Let

$$\epsilon_\theta = -\lim_{L,T\to\infty}\frac{1}{LT}\ln\left(Z^\theta_{L\times T}\right) \tag{5.9}$$

denote the vacuum energy density, normalized so that $\epsilon_0 = 0$. [The existence of the limit on the right-hand side of (5.9) is a standard consequence of reflection positivity.] The function $\epsilon_\theta$ is the analog of $P_\theta(\beta,z)$. It is symmetric about $\theta = e$, a property it shares with $P_\theta(\beta,z)$.

Let $E$ be the real-time electric field. Then

$$-i\langle F(x)\rangle_\theta = \langle E(x)\rangle_\theta = 2\frac{d}{d\theta}\epsilon_\theta \tag{5.10}$$

see Refs. 22, 19, and 21. We note that equation (5.10) corresponds to equation (3.11) for the one-dimensional Coulomb gas. Analogously to Proposition 4 exists the following proposition.

**Proposition 5.**

$$\left\langle \exp\left[i\alpha\oint_{\mathcal{C}_{L\times T}} A_\mu(\xi)\,d\xi^\mu\right]\right\rangle_\theta \leqslant \exp\{-4LT[\epsilon_{\theta+2\alpha}-\epsilon_\theta]\}$$

This inequality was found in Ref. 23 and, independently, in Ref. 21. The proof is a standard consequence of reflection positivity and is very similar to the one of Proposition 4. Proposition 5 shows that if $\epsilon_{\theta+2\alpha} > \epsilon_\theta$ then the expectation of the *Wilson loop*, $\exp[i\alpha\oint_{\mathcal{C}_{L\times T}} A_\mu(\xi)\,d\xi^\mu]$, in the $\theta$ vacuum has what is called *area decay*. The physical interpretation of this fact is that two static sources of fractional charge $\pm\alpha$, separated by a distance $2L$, feel a constant, attractive force with potential $2[\epsilon_{\theta+2\alpha}-\epsilon_\theta]\cdot L$.

It is obvious from the definition of $\epsilon_\theta$ that $\epsilon_\theta \geqslant \epsilon_0 = 0$, for all $\theta \in (0, 2e)$. In fact, by a result of Brydges *et al.*[12] which we recall below,

$$\epsilon_\theta > \epsilon_0 \qquad \text{for} \quad 0 < \theta \leqslant e \tag{5.11}$$

for the two-dimensional lattice Higgs model. Thus, in the state $\langle - \rangle_{\theta=0}$, static, fractionally charged sources are permanently confined. On the other hand, for $\theta = e$, $\epsilon_{\theta+2\alpha} < \epsilon_\theta$, thus the sources repel each other by a constant force; see, e.g., Ref. 24. In other states both types of behavior may occur, depending on the value of $\alpha$, in analogy with the situation met in the one-dimensional Coulomb gas described in Section 4. Finally, we mentioned a particularity of the quantum field theories which is *not* found in the Coulomb gas: At $\theta = e$ a phase transition may occur, as the values of the coupling constants of the theory, in particular the electric charge, are varied. There may exist two distinct vacua, $\langle - \rangle_{e+}$ and $\langle - \rangle_{e-}$, with

$$\langle E(x) \rangle_{e+} = -\langle E(x) \rangle_{e-} \neq 0 \tag{5.12}$$

This corresponds to $\epsilon_\theta$ having a discontinuous first derivative at $\theta = e$. The theories determined by the states $\langle - \rangle_{e\pm}$ have a *charged* (*soliton*) *sector*. Proofs of the existence of that transition and of soliton sectors in a variety of two-dimensional models can be found in Ref. 25; see also Ref. 12.

We conclude this section by recalling a *connection* between two-dimensional lattice Higgs models and the one-dimensional lattice Coulomb gas found in Ref. 12. In that reference the following inequality is proven for the corresponding lattice models:

$$\left\langle \exp\left( i\alpha \oint_{\mathcal{C}_{L\times T}} A_\mu(\xi)\, d\xi^\mu \right) \right\rangle_{\theta=0} \leqslant \left\{ \left\langle e^{-i\alpha\phi(L)} e^{i\alpha\phi(-L)} \right\rangle \right\}^T \tag{5.13}$$

the right-hand side of (5.13) being the fractional charge correlation in the lattice Coulomb gas. The proof consists of applying correlation inequalities of the Ginibre type; see Ref. 12. Inequality (5.13) together with the results of Section 4 imply area decay of the Wilson loop expectation in the $\theta = 0$ vacuum for *arbitrary* values of the coupling constants. By using (5.13) to study the asymptotics of $\langle \exp(i\alpha \oint_{\mathcal{C}_{L\times T}} A_\mu(\xi)\, d\xi^\mu) \rangle_{\theta=0}$, as $L, T \to \infty$, and applying Proposition 5 we obtain inequality (5.11), as announced. For results in four-dimensional, non-Abelian Yang–Mills theories see, e.g., Refs. 4, 26.

## 6. CONCLUSIONS, OPEN PROBLEMS

In this paper we have obtained results on the structure of equilibrium states and the behavior of fractional charge correlations in the one-dimensional Coulomb gas. These results are the analog of well-known results concerning $\theta$ vacua and confinement of fractional charges in models of two-dimensional QED (and four-dimensional non-Abelian Yang–Mills theories). It is the point of this note to summarize and compare those two circles of ideas and results, in a pedagogical and rather pedestrian way. We hope that this may contribute a grain of stimulation for field theorists and statistical mechanicians to follow each other's ideas with an open mind. Since the reader may have gained the impression that one-dimensional Coulomb gases (and two-dimensional Abelian gauge theories) are, by now, overstudied topics, we conclude with a short selection of comments and open problems, designed to convince him or her that this need not be quite so. (1) It is amusing to note that the theory of the one-dimensional Bloch electron, in its functional integral (imaginary-time) version isomorphic to the one-dimensional two-component Coulomb gas, is also isomorphic to the quantum mechanics of the spherical pendulum in a constant gravitational field. This connection is obvious for $\theta = 0$. The theory for $\theta \neq 0$ corresponds to quantizing the pendulum in a rotating frame corresponding to a gravitational field that rotates with uniform angular velocity $\theta$. The reader may verify this as an exercise. (We thank G. Gallavotti for drawing our attention to this example.) (2) One may study a Coulomb gas consisting of $2n$ species of particles with charges $\pm me$ and activities $z_m$, $m = 1, 2, \ldots, n$, $n \leqslant \infty$. The study of this system is equivalent to that of the spectral properties of Hill's operator,

$$-\beta^{-1}\frac{d^2}{du^2} + V(u)$$
$$V(u) = 2\sum_{m=1}^{n} z_m \cos(meu) \tag{6.1}$$

The equivalence is seen in the same manner as the relation between the standard two-component Coulomb gas and the Mathieu equation; see Sections 2 and 3. The operator defined in (6.1) has been studied in great detail in Ref. 27. The results for the generalized Coulomb gas emerging from that reference, and the various phenomena, are qualitatively the same as the ones for the two-component gas. For this reason we omit its further discussion. (3) More interesting is the study of an "exotic" Coulomb gas of $2n$ species of particles with changes $\pm e_m$ and activities $z_m$, $m = 1, \ldots, n$, $n < \infty$, where some of the charges are *irrationally related.* As in Sections 2 and 3 one shows that the theory of this gas is isomorphic to the functional integral (imaginary-time) formulation of the quantum mechanics of a nonrelativistic particle of mass $\beta/2$ in a quasiperiodic potential, $V(u)$, given by

$$V(u) = 2\sum_{m=1}^{n} z_m \cos(e_m u) \tag{6.2}$$

The Schrödinger Hamiltonian of this system is

$$H \equiv -\beta^{-1}\frac{d^2}{du^2} + V(u) \tag{6.3}$$

Under suitable assumptions on $\{z_m\}$ and $\{e_m\}$, Dinaburg and Sinai[28] and Rüssmann[29] have shown that the spectrum of $H$ has a band structure. In that region of activities and charges, the qualitative behavior of the corresponding Coulomb gas is similar to the standard two-component gas. (In particular, there is a family of equilibrium states, "confinement" of certain charges and no exponential Debye screening, as discussed in Section 4.) It might happen, however, that for a suitable choice of $\{e_m\}$ and sufficiently large activities, $z_m$, screening sets in. This could be so, because external sources with arbitrary charge $\alpha$ can possibly be screened by particles in the ensemble, since $\alpha$ can be approximated by numbers $\sum_{m=1}^{n} k_m e_m$, $k_m \in \mathbb{Z}$. Thus, if the source with charge $\alpha$ is surrouned by $k_m$ particles of charge $-e_m$, $m = 1, \ldots, n$, its charge is screened almost completely. Such configurations might be quite likely if the activities are large enough and if the approximation of $\alpha$ by $\sum_{m=1}^{n} k_m e_m$ is fast in the sense that the error becomes quite small already for "small" values of $\sum_{m\neq 0}|k_m|$. Screening in this exotic Coulomb gas would mean that the associated Schrödinger operator introduced in (6.2), (6.3) has an eigenvalue at the bottom of its spectrum. This speculation, and hence the existence of phase transitions as $\beta$, $\{e_m\}$, and $\{z_m\}$ are varied, are somewhat supported by recent results of Aubry.[30] (We thank H. Kunz for an instructive discussion of this point.)

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